R. Perry
14 Nov. 1978

THE TAU METHOD

Problem #35, page 329 of A First Course in Numerical Analysis by Ralston and Rabinowitz.
P670 Numerical Analysis, Prof. Fair

***35** The τ method. Consider the differential equation

(1) $$L(y) = p_n(x)\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_0(x)y + P(x) = 0$$

(2) $$y(0) = y_0 \qquad y^{(i)}(0) = y_i \qquad i = 1, \ldots, n-1$$

where $p_j(x)$ is a polynomial of degree d_j and $P(x)$ is a polynomial of degree d.

(*a*) Assume a solution

(3) $$y(x) = \sum_{j=0}^{m} a_j x^j \qquad m \ge n$$

and let

(4) $$D = \max(d, d_0 + m, d_1 + m - 1, \ldots, d_n + m - n)$$

Show that substituting this solution into the differential equation leads in general to a system of $D + 1 + n$ equations in $m + 1$ unknowns if the initial conditions are to be satisfied.

(*b*) Show that, in general, the differential equation

(5) $$L(y) = \sum_{i=1}^{D-m+n} \tau_i T^*_{m-n+i}(x)$$

does have a solution of the form of part (*a*), where the τ_i's are real numbers and T^*_{m-n+i} is the shifted Chebyshev polynomial of degree $m - n + i$ (see Prob. 23). Thus deduce that if the τ_i's are small, the solution of this differential equation is a good approximation to the solution of $L(y) = 0$.

(*c*) Use this method with $m = 4$ to approximate e^x. Draw a graph of the error on [0, 1] and compare this with the error in $R_{4,0}(x)$ on [0, 1]. [This method is of particular value when $f(x)$ does not have a convergent polynomial or continued-fraction expansion.] [Ref.: Lanczos (1956), pp. 464–469.]

Differentiation of (3) yields:

(6) $$y^{(k)}(x) = \sum_{j=0}^{m} j(j-1)\cdots(j-k+1)a_j x^{j-k}$$

Substitution of the boundary conditions (2) into equations (3) and (6) yields:

$$a_0 = y_0 \qquad a_1 = y_1 \qquad a_2 = y_2/2 \qquad \ldots\ldots \qquad a_{n-1} = y_{n-1}/(n-1)!$$

Thus n constants, a_0 to a_{n-1}, are determined by the initial conditions, and (m-n+1) unknowns, a_n to a_m, remain to be determined.

Note that the maximum degree of x in y is m, in $y^{(1)}$ is m-1, ..., in $y^{(n)}$ is m-n. So using (4), D is the maximum degree of x in equation (1). Since L(y) must equal zero, the coefficient of each power of x, from x^0 to x^D, must equal zero. This gives D+1 equations in m-n+1 unknowns, a_n to a_m. Since the minimum value of D is m, there will always be more equations than unknowns and in general this overdetermined system will not have a solution.

Overall, there were (D+1)+n equations with (m-n+1)+n=m+1 unknowns, a_0 to a_m.

When we substitute a finite power series for y into equation (1), it is generally impossible to solve the resulting overdetermined system. Consider solving (5) instead, where zero has been replaced by a sum of shifted Chebyshev polynomials of the proper degree. Since the shifted Chebyshev polynomials have the min-max property over the range x from 0 to 1, and have a peak value of 1, the peak value of the right hand side of (5) will be less than or equal to: $\sum_{i=1}^{D-m+n} |\tau_i|$.

Therefore if the τ_i's are small, then $L(y) \approx 0$ and the solution of (5) is a good approximation to the solution of (1).

From (5), we have a sum of shifted Chebyshev polynomials, T^*_D , T^*_{D-1} , ... , T^*_{m-n+1} , with (D-m+n) unknowns, $\tau_1 \cdots \tau_{D-m+n}$. Note that the highest degree of x is D. After solving the initial conditions, substituting (3) into (5) now yields (D+1) equations in (m-n+1)+(D-m+n)=D+1 unknowns, and thus in general a solution will exist.

Example e^x satisfies the differential equation $L(y)=y-y'=0$.

let $y= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

then $y'= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$.

The boundary condition $y(0)=1$ yields $a_0=1$.

Note that m=4, n=1, D=max(0,0+m,0+m-1)=m=4, D-m+n=1,

and the error term is τT^*_4 .

Equation (5) becomes in this case:

$$L(y)= y-y'=(1-a_1) + (a_1-2a_2)x + (a_2-3a_3)x^2 + (a_3-4a_4)x^3 + a_4x^4 = \ldots$$
$$\ldots = \tau[\,1 \;\; -32x \;\; + 160x^2 \;\; - 256x^3 \;\; + 128x^4\,] .$$

where $T^*_4 = 1 - 32x + 160x^2 - 256x^3 + 128x^4$. (The shifted polynomial T^*_n can be derived from T_n by a change of variables, $T^*_n(x)=T_n(2x-1)$.)

Equate powers of x on the left and right hand sides:

$$\left.\begin{aligned} 1-a_1 &= \tau \\ a_1-2a_2 &= -32\tau \\ a_2-3a_3 &= 160\tau \\ a_3-4a_4 &= -256\tau \\ a_4 &= 128\tau \end{aligned}\right\}$$

This system of five equations in five unknowns can be solved by hand using Gaussian elimination and back substitution with exact arithmetic. The solution is :

a_1= 1824/1825. τ = 1/1825.= .548 E-3

a_2= 928/1825.

a_3= 256/1825.

a_4= 128/1825.

Thus, $y(x) = 1 + \frac{1824}{1825} x + \frac{928}{1825} x^2 + \frac{256}{1825} x^3 + \frac{128}{1825} x^4$.

Compare this expression with the fourth-order Taylor series approximation to e^x :

$$R_{4,0}(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 .$$

Table 1 shows the results of a Fortran program run to compare e^x with the approximation , $R_{4,0}(x)$,(Taylor series),and with the Tau approximation,Y(x). The errors of each are plotted in figure 1.

The Taylor series approximation has less error than the Tau method for x between 0 and .35 . This is to be expected, since the Taylor series approximation satisfies five boundary conditions at x=0 , and is most accurate near that point. The Tau approximation has an error bounded by .98 E-4, for x between .4 and 1 , but the error in $R_{4,0}$ increases by two orders of magnitude beyond this over the same interval.

(The anomalous point ER=0.0 at x=.1 should be recomputed using double precision)

```
FORTRAN IV      V01C-03D       SUN 12-NOV-78 12:22:17          PAGE 001
CORE=08K, UIC=[200,205]                                        ,TTO:=TAU

           X=0.0
           DO 1 I=1,20
           XI=I
           X=XI*0.05
           R=(X*(X*(X*(X+4.)+12.)+24.)+24.)/24.
           Y=(X*(X*(X*(128.*X+256.)+928.)+1824.)+1825.)/1825.
           Z=EXP(X)
           ER=R-Z
           EY=Y-Z
     1     WRITE(6,2) X,Z,R,Y,ER,EY
     2     FORMAT(1X,F6.2,3F16.9,2E20.9)
           STOP
           END
```

R. Perry

X	e^x	R	Y	ER	EY
0.05	1.051271200	1.051271081	1.051261783	-0.119209290E-06	-0.941753387E-05
0.10	1.105170846	1.105170846	1.105177402	0.000000000E+00	0.655651093E-05
0.15	1.161834359	1.161833644	1.161867857	-0.715255737E-06	0.334978104E-04
0.20	1.221402884	1.221400023	1.221464634	-0.286102295E-05	0.617504120E-04
0.25	1.284025311	1.284016967	1.284109592	-0.834465027E-05	0.842809677E-04
0.30	1.349858761	1.349837422	1.349955559	-0.213384628E-04	0.967979431E-04
0.35	1.419067621	1.419021130	1.419165373	-0.464916229E-04	0.977516174E-04
0.40	1.491824746	1.491733432	1.491912842	-0.913143158E-04	0.880956650E-04
0.45	1.568312407	1.568146229	1.568381906	-0.166177750E-03	0.694990158E-04
0.50	1.648721337	1.648437500	1.648767114	-0.283837318E-03	0.457763672E-04
0.55	1.733252764	1.732792020	1.733273864	-0.460743904E-03	0.211000443E-04
0.60	1.822118998	1.821400166	1.822117805	-0.718832016E-03	-0.119209290E-05
0.65	1.915540695	1.914458752	1.915524960	-0.108194351E-02	-0.157356262E-04
0.70	2.013752699	2.012170792	2.013731956	-0.158190727E-02	-0.207424164E-04
0.75	2.117000103	2.114746094	2.116986275	-0.225400925E-02	-0.138282776E-04
0.80	2.225541115	2.222399950	2.225545645	-0.314116478E-02	0.452995300E-05
0.85	2.339647293	2.335354567	2.339678288	-0.429272652E-02	0.309944153E-04
0.90	2.459603310	2.453837633	2.459663153	-0.576567650E-02	0.598430634E-04
0.95	2.585709810	2.578083515	2.585788965	-0.762629509E-02	0.791549683E-04
1.00	2.718281984	2.708333254	2.718356133	-0.994873047E-02	0.741481781E-04

Table 1

R. Perry
12 Nov. 78

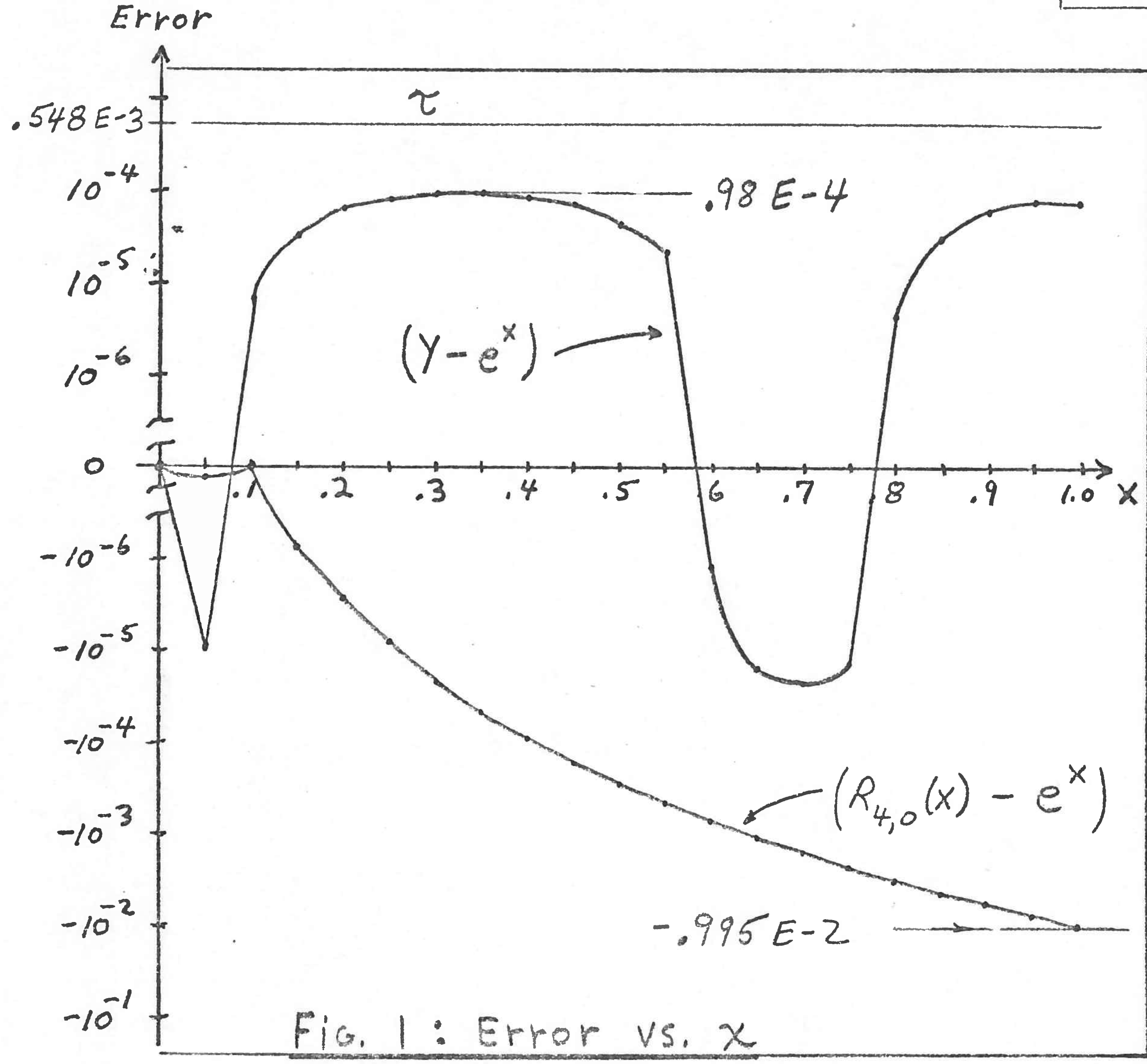

Fig. 1: Error vs. x

Taylor Series Approximation to e^x:

$$R_{4,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

TAU Method to Approximate e^x:

$$Y(x) = 1 + \frac{1824}{1825}x + \frac{928}{1825}x^2 + \frac{256}{1825}x^3 + \frac{128}{1825}x^4.$$