Interference and Quantum Computations

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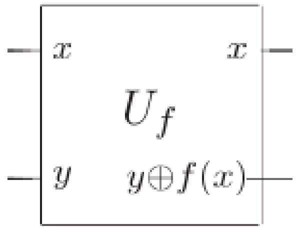
Quantum computation of a function *f(x)* is generally described by a unitary transformation U_f such that:

$$U_f\,|x, y\rangle = |x, y \oplus f(x)\rangle$$

corresponding to the circuit diagram ([1], page 31):



These descriptions are correct if $x,y \in \{0,1\}$. But if *x* and/or *y* are superpositions then these descriptions may be misleading, and in particular the input *x* may not pass through U_f unchanged due to interference between the qubits.

Note that the ⊕ operation and function *f(x)* can only be applied directly to *{0,1}* operand values. For superpositions the operations are applied separately to each part.

In general we have:

$$U_f\,|x, y\rangle = |\Psi\rangle$$

$$= |w, z\rangle \quad \textit{if separable}$$

Consider the CNOT operation, so $f(x) = x$ and U_f performs $y \oplus x$.

If $|x\rangle = |0\rangle + |1\rangle$ and $|y\rangle = |0\rangle - |1\rangle$ then (leaving out some factors of 1/sqrt(2)):

$$\text{CNOT}\,|x, y\rangle = |0\rangle (|0\oplus0\rangle - |1\oplus0\rangle) + |1\rangle (|0\oplus1\rangle - |1\oplus1\rangle)$$

$$= |0\rangle (|0\rangle - |1\rangle) + |1\rangle (|1\rangle - |0\rangle)$$

$$= |0\rangle (|0\rangle - |1\rangle) - |1\rangle (|0\rangle - |1\rangle)$$

$$= (|0\rangle - |1\rangle) (|0\rangle - |1\rangle)$$

$$= |w, y\rangle$$

So $|x\rangle$ is changed to $|w\rangle = |0\rangle - |1\rangle$ and *y* is unchanged.

In this case the action of operating on *y* with *f(x)* changes *x* and does not affect *y*.

This effect is due only to interference between the qubits. There is no entanglement as the result is separable into a product state.

For comparison, entanglement would occur using inputs |*x*> = |*0*> + |*1*> and |*y*> = |*0*>:

CNOT |x, y> = |0> |0⊕0> + |1> |0⊕1>

= |0> |0> + |1> |1>

The state |*00*> + |*11*> is entangled and can not be factored into a product state.

From the viewpoint of a quantum computing simulation, *x*,*y* are indices into an array of amplitudes.

An amplitude table for |*w*,*z*> = CNOT |*x*,*y*> is constructed by setting *w*=*x* and *z*=*x*⊕*y*.

Leaving out some factors of 1/sqrt(2), the initialization |*x*> = |*0*> + |*1*> and |*y*> = |*0*> - |*1*> corresponds to:

(|0> + |1>) (|0> - |1>) = |00> - |01> + |10> - |11>

Letting *a* represent amplitude in the table, with input amplitudes *q* and output amplitudes *r*:

```
                CNOT: z = x⊕y

         x y   a  -> x z   a  =  x y   a
q[0]     0 0   1     0 0   1     0 0   1      r[0] = q[0]
q[1]     0 1  -1     0 1  -1     0 1  -1      r[1] = q[1]
q[2]     1 0   1     1 1   1  \/ 1 0  -1      r[2] = q[3]
q[3]     1 1  -1     1 0  -1  /\ 1 1   1      r[3] = q[2]
```

Effectively, CNOT simply swaps the amplitudes of states |*10*> and |*11*>.

Although the table was constructed by explicitly replacing *y* with *x*⊕*y*, the result is the same as described above for interference, where *x* is changed and *y* is unchanged, which has amplitudes:

(|0> - |1>) (|0> - |1>) = |00> - |01> - |10> + |11>

For comparison, the amplitude table for entanglement, with initialization |*x*> = |*0*> + |*1*> and |*y*> = |*0*>, is:

```
                CNOT: z = x⊕y

         x y   a  -> x z   a  =  x y   a
q[0]     0 0   1     0 0   1     0 0   1      r[0] = q[0]
q[1]     0 1   0     0 1   0     0 1   0      r[1] = q[1]
q[2]     1 0   1     1 1   1  \/ 1 0   0      r[2] = q[3]
q[3]     1 1   0     1 0   0  /\ 1 1   1      r[3] = q[2]
```

References

[1] Quantum Computation and Quantum Information, 10th Anniversary Edition, Michael A. Nielsen & Isaac L. Chuang, Cambridge University Press, 2010.